Logique & Analyse **255** (2021), 245-265

ACHILLES AND THE TORTOISE 2500 YEARS AFTER ZENO

Theo Theunissen and Johan H.L. Oud

Abstract

After a short introduction to the well-known paradox of Achilles and the tortoise, originating from the 5th century B.C. and attributed by both Plato and Aristotle to Zeno of Elea, we present an overview of important solutions and discussion themes of the paradox, using only basic mathematics. In addition to the limit solution, based on Cauchy's limit concept, a simpler but less well-known geometric solution is provided. This yields not only the exact distance but also the exact amount of time that Achilles needs to pass the tortoise. Subsequently, we discuss the philosophical background of the paradox. In particular, what Zeno wanted to prove with the paradox and how Aristotle in his criticism foreshadowed both the limit and the geometric solution. Next, the paradox is placed in the perspective of modern quantum theory. This leads to the replacement of classic continuous spacetime by 'granular' spacetime and to a new discrete solution of the paradox in terms of granular spacetime. Finally, the problem of supertasks in addition to other topics related to the paradox are dealt with.

Keywords: continuous spacetime, discrete solution, geometric solution, granular spacetime, limit solution, passing the tortoise, quantum theory

1. Introduction

The paradox of Achilles and the tortoise is one of several paradoxes formulated by Zeno of Elea in the 5th century B.C. in support of the doctrine of his master Parmenides and it is Zeno's most well-known paradox. It features a footrace between Achilles, a popular runner in antiquity, and a tortoise. The essence of the paradox is that, if Achilles allows the tortoise a certain head start, he will never be able to catch up with it. Each time Achilles covers the distance between himself and the tortoise, the tortoise has already run a new distance that should next be covered by Achilles. This cycle can be repeated an infinite number of times, with the result that Achilles will never pass the tortoise. Until way beyond the Middle Ages, the cleverest people dedicated their best efforts to bridge this infinity gap. It took almost 2300 years before the paradox was satisfactorily solved by means of the limit concept, which Augustin-Louis Cauchy used to lend a firm basis for Newton's and Leibniz's calculus.

doi: 10.2143/LEA.255.0.0000000

The purpose of the article is to present essential solutions and discussion themes of the paradox, using basic mathematics only. Because the literature about Zeno and his paradoxes is immense and every year no less than 150 new publications are added, no attempt can be and is made to be complete. In addition to a simple version of the limit solution, a second geometric solution is presented, which is even simpler and enables one not only to calculate the exact distance but also the exact time Achilles needs to catch up with the tortoise. The relationship between these two historic solutions is clarified and the philosophical background of the paradox discussed, in particular the way Aristotle handled the paradox. Next, we consider the paradox from the perspective of modern quantum theory, resulting in a third discrete solution, which instead of in classic continuous spacetime is staged in so-called 'granular' spacetime. This and the popular topics of supertasks and infinity machines in the final section show how relevant Zeno's thinking still is for modern physics and philosophy. Bertrand Russell evaluated in 1914: "Zeno's arguments, in some form, have afforded grounds for almost all the theories of space and time and infinity which have been constructed from his day to our own" (Russell 1914, p. 54). No doubt Russell's words will continue to be true for many years to come.

2. Historic solutions in continuous spacetime

2.1. *Limit solution*

To analyze Zeno's reasoning and where it missed the point, a few simple arithmetic facts need to be considered first. It is quite straightforward that the sequence of numbers

$$1,\ v,\ v^2,\ v^3,\ \ldots,\ v^{n-1} \tag{1}$$

(e.g., 1, 3, 9, 27, 81 for $v = 3$ and $n = 5$), when added, results in the series

$$1 + v + v^2 + v^3 + \ldots + v^{n-1} = \frac{1 - v^n}{1 - v} \tag{2}$$

(multiply the sum by $1 - v$ to find $1 - v^n$) and next leads to

$$s + sv + sv^2 + sv^3 + \ldots + sv^{n-1} = s\,\frac{1 - v^n}{1 - v} \tag{3}$$

To solve the paradox we need only one more result. If we allow n to go to infinity ($n \to \infty$), this quantity converges to limit

$$s\,\frac{1}{1 - v} \tag{4}$$

for all values of $v < 1$. If v is smaller than 1, then v^n approaches 0, allowing us to write $s\frac{1}{1-v}$ as limiting value of $s\frac{1-v^n}{1-v}$. The term limit indicates that the limiting value is not actually reached but only approximated, in steps with $s\frac{1-v^n}{1-v}$ becoming increasingly closer to $s\frac{1}{1-v}$ (expressed as $s\frac{1-v^n}{1-v} \to s\frac{1}{1-v}$) as we let n increase stepwise to infinity ($n \to \infty$).

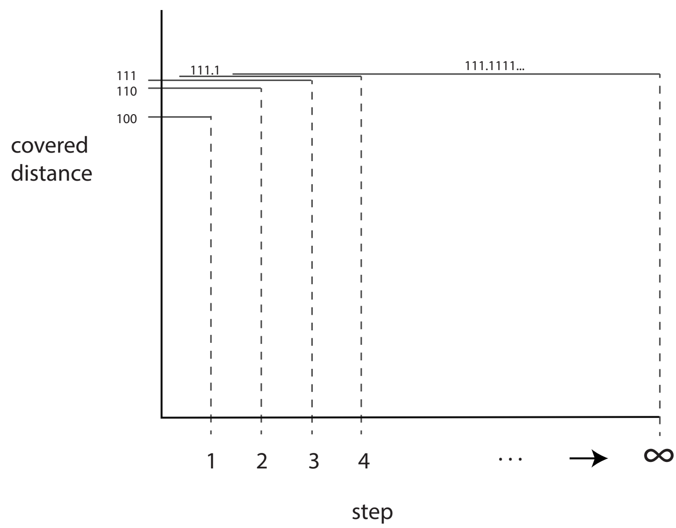


Figure 1: In the limit solution the number of added head-starts increases in steps to ∞ but the total distance covered cannot exceed $s/(1 - v)$ = 111.1111 … meters. This is the distance at which Achilles catches up with the tortoise in the example with s = 100 meters as the first head-start of the tortoise and with its speed being v = 0.1 of Achilles' speed. Starting with 100 meters in step 1, each following step adds a 1 before or after the decimal point but this keeps the total within the limit of 111.1111 … meters.

Now, let us apply the arithmetic facts in the previous paragraph to solve the paradox. Suppose that the tortoise only runs $v < 1$ times as fast as Achilles and Achilles therefore runs $1/v$ times as fast as the tortoise (e.g., the tortoise having only $v = 0.1$ of the speed of Achilles and, consequently, Achilles is $1/v = 10$ times as fast as the tortoise). In addition, at the start, the tortoise has a head start of $s = 100$ meters. When Achilles covers the $s = 100$ meters distance, the head start of the tortoise shrinks to $sv = 10$ meters, and when Achilles next covers this, shrinkage is to $sv^2 = 1$ meters and so on. We can go on until infinity. Added in the series, however, according to the limit above, the head starts in total cannot exceed $s\frac{1}{1-v} = 100\frac{1}{1-0.1}$ = 111.1111 … meters. In each step a 1 before or after the comma is added but clearly the total will never exceed 112 meters. Therefore, Achilles will pass the tortoise in less than 112 meters, in contrast to what Zeno of Elea claims (see Figure 1). His paradox mistakenly interprets n as time. However, n does not stand for a fixed interval of time and does not go to an infinitely far away point in time. It increases in steps to ∞ over a relatively short, calculable period of time. The number of steps goes to infinity, but neither the distance nor the elapsed time does. The next solution will yield both distance and period of time required.

2.2. *Geometric solution*

The geometric solution specifies two simple linear equations: one for Achilles and one for the tortoise. If we assume that Achilles maintains a constant speed and, for the moment, we take 1 time unit as the time he needs to cover 1 distance unit (1 meter in the example above), then the period of time becomes $s\frac{1}{1-v}$. For example, when the time unit is 1 second, then it becomes $s\frac{1}{1-v}$ seconds as a result. This can be seen as follows. Let p_A in the first equation for Achilles

$$p_A = t \tag{5}$$

be the position reached by Achilles during the time interval t (as many distance units as time units), and p_T in the second equation for the tortoise

$$p_T = s + vt \tag{6}$$

the position reached by the tortoise during the same time interval. Clearly, the tortoise has a head start s and runs only v times as fast as Achilles, that is, covers only vt during the time Achilles covers t. To find the period t they need to reach the same position, we equate the positions and call this same position $p : p_A = p_T = p$. In the first equation, this produces $p = t$, and in the second equation, because $p_T = p = t$ (the position of the tortoise being equal to the one of Achilles which is t according to the first equation), $t = s + vt$ or $(1 - v)\, t = s$. Therefore, the period of time required becomes $t = s\frac{1}{1-v}$, that is, equal to the distance that Achilles needed in the limit solution and is needed also in the present solution to pass the tortoise. If it takes Achilles 1 second to cover 1 meter, then he passes the tortoise in 111.1111 … seconds, that is slightly less than 1.852 minutes.

Suppose, however, it takes him only half a second to cover 1 meter, then the time required becomes half this value, namely 0.926 minutes. In general, the period of time can be written as

$$s\frac{1}{1-v}b \tag{7}$$

if Achilles takes b time units to cover 1 distance unit. See Figure 2 for a graphical representation. It should be noted that in the case of $b \neq 1$, where Achilles needs $b \neq 1$ time units to cover 1 distance unit, time should first be divided by b to determine the distance covered: t/b instead of t. Because b is the number of time units needed per distance unit, $1/b$ or the number of distance units per time unit is Achilles' speed. v/b is the speed of the tortoise, because the tortoise is v times as fast as Achilles. Note also that for the calculation of the crossing point between the two linear curves, $s\frac{1}{1-v}$, that is the distance at which they meet, the value of b does not matter, because filling in time period $s\frac{1}{1-v}b$ for t in the equations $p = t/b$ and $p = s + v(t/b)$ results in $p = s\frac{1}{1-v}$ once again. So, it is only the head start s and the relative speed v between both that determine the meeting distance.

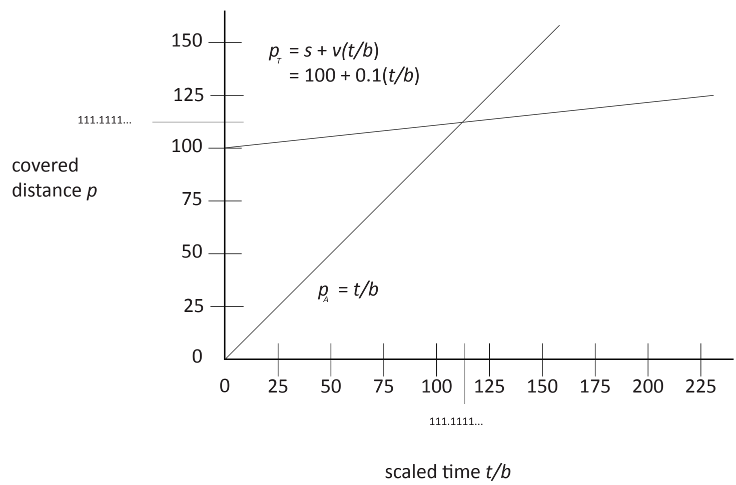


Figure 2: In the geometric solution, Achilles and the tortoise each have their own equation (linear curve) for the relationship between time and distance covered, which cross at the point where they pass: $s/(1 - v) = 111.1111 \ldots$ (for $s = 100$ and $v = 0.1$). This point is reached at the end of time period $[s/(1 - v)]b$, where b is the number of time units Achilles needs to cover 1 distance unit.

The geometric solution by means of the two linear equations $p = t$ and $p = s + vt$ or, in case $b \neq 1$, $p = (1/b)t$ and $p = s + (v/b)t$ in two unknowns (p and t) is simpler than the series approach used in the limit solution. Both show where Achilles catches up with the tortoise, but the series approach reveals more clearly where the argumentation of the paradox goes wrong. Specifically, the number of steps in Zeno's reasoning indeed goes to infinity (increases without bound), but the result nonetheless remains within a finite distance and period of time. An important difference between the two solutions is that the crossing point of the two curves in Figure 2 is a specific value that can be calculated directly, while a limit is mathematically more complex. Every *n* corresponds to one specific value of the series and every higher *n* corresponds to a value closer to the limit, but for the very limit itself there is no *n* to calculate the value.

2.3. *Putting limit and geometric solution together*

Although both solutions are mathematically different, by combining them Figure 3 clarifies their relationship and provides a detailed picture of how the distance between Achilles and the tortoise develops across time. p_T and p_A in Figure 2 and Figure 3 are continuous variables, meaning that the differences between the values on both the p_T-scale and the p_A-scale can be regarded as infinitely small. If we start with the head start $p_T(0) = s$ of the tortoise, which we let Achilles run in step 1, $p_A(1) = p_T(0) = s$, we obtain

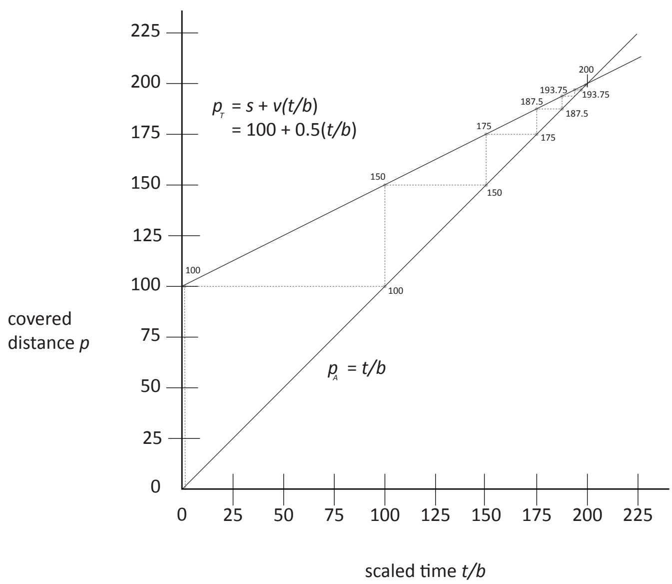


Figure 3: In the combined solution (shown here for the example $s = 100$, $v = 0.5$), the crossing point 200 of the linear curves p_A and p_T of Achilles and the tortoise is also the limit of the distance which each of them runs in steps. The distances indicated on the curves are found by alternating $p_A(i) = p_T(i-1)$ and $p_T(i) = s + vp_A(i)$: in each step Achilles runs what the tortoise has run in the previous step (horizontal curves) and the tortoise runs his head start s plus, in lower speed v, what Achilles has run. The vertical curves provide the successive distance differences between tortoise and Achilles (head starts of the tortoise): $s = 100$ at the very beginning and next $sv = 50$, $sv^2 = 25$, $sv^3 = 12.5$, $sv^4 = 6.25$, etc. The difference reduces to 0 when approaching limit 200.

for the tortoise in step 1 the distance $p_T(1) = s + vp_A(1) = s + vs$. We then put this value on the p_A-scale again: $p_A(2) = p_T(1)$. By alternating

$$p_A(i) = p_T(i - 1) \tag{8}$$

and

$$p_T(i) = s + vp_A(i) \tag{9}$$

we find as successive differences between the values $p_T(i)$ and $p_A(i)$ on both scales

$$s,\ vs,\ v^2 s,\ v^3 s,\ \ldots,\ v^{n-1} s \tag{10}$$

Thus

$$p_T(i) - p_A(i) = p_A(i + 1) - p_A(i) = p_T(i) - p_T(i - 1) = v^i s \tag{11}$$

and added across time we find as shown earlier

$$s + vs + v^2 s + v^3 s + \ldots + v^{n-1} s = s\frac{1 - v^n}{1 - v} \tag{12}$$

The terms $s,\ vs,\ v^2s,\ v^3s,\ \ldots,\ v^{n-1}s$ in the series become smaller and smaller when approaching the crossing point. It is only at the crossing point that we find exactly the same values, $p_A = p_T$, at one and the same point in time. Before the crossing point, we write $p_A(i) = p_T(i - 1)$, where i and $i - 1$ are different steps at different points in time, while $p_T(i)$ and $p_A(i)$ in each step $i < \infty$ take on different values according to $p_T(i) = s + vp_A(i)$. The elapsed time t in each step i is calculated as $t(i) = bp_A(i) = b\frac{p_T(i) - s}{v}$, resulting for $p_A = p_T = p$ at the crossing point into $t = s\frac{1}{1 - v}b$ as found above in equation (7). For Achilles and the tortoise in each step we find $p_A(i) = s\frac{1 - v^i}{1 - v}$ and $p_T(i) = s\frac{1 - v^{i+1}}{1 - v}$.

To illustrate the combined solution adequately, we choose an example with relatively smaller distances between the steps, namely: $s = 100$ and $v = 0.5$ instead of 0.1. For this example, Figure 3 shows that the distance values p_A and p_T, calculated stepwise, are 100, 150, 175, 187.5, 193.75, …, respectively. When plotted on the curves in Figure 3, it becomes clear that the values come closer and closer to the crossing point 100/(1−0.5) = 200 but do not exceed it, because the distances between the values on the p_A and p_T scales, when approaching the crossing point, reduce to zero. In the combined solution, each value in the series is used twice, once for the tortoise and once for Achilles, but at different points in time. Only at the crossing point (in the limit) the same value is found at the same point in time.

The crossing point

$$p_A = p_T = s\frac{1}{1-v} \tag{13}$$

only exists, that is, Achilles and the tortoise will only be able to meet, if $v < 1$. Values $v \geq 1$ would imply that the curves in Figures 2 and 3 run parallel or diverge, which would occur when the runner with the head start would run as fast or faster than the one who lags behind. Thus, the condition $v < 1$ is part of both solutions. The limit method in the combined solution, where the values are alternately placed on the p_A-scale and the p_T-scale and approach the crossing point in steps, is called by Ishikawa (2015) the iterative calculation of the crossing point, thus distinguishing it from the direct, algebraic calculation presented in the geometric solution. For many problems we have no choice but to resort to the iterative or limit solution, because there is no direct one. Apparently, this is not true for our case of Achilles and the tortoise, where we have two solutions that both lead to exactly the same result.

3. Philosophical background of the paradox

The question remains what Zeno wanted to prove with the paradox of Achilles and the tortoise and his other paradoxes, because obviously he was also aware of the fact that Achilles would pass the tortoise. Unfortunately, none of Zeno's writings has survived. He is introduced by Plato in his *Parmenides* as a participant in a dialogue with three other well-known philosophers in antiquity: Parmenides, Socrates and Aristotle (Plato 1997). In addition, Aristotle mentions Zeno in his *Physica*, where he tries to refute Zeno's arguments (Aristotle 1957). This was the time when philosophers discussed the theme of the One versus the Many. And it was the unity of Being, the basis of Plato's theory of ideas, that made him invite Parmenides and Zeno to participate in the dialogue. Plato tells us that Parmenides and Zeno were not only lovers but also partners in defense of the One.

In contrast to Zeno's writings, some text of Parmenides has survived. In his poem *On nature*, Parmenides discusses the problem of the One and the Many in a particularly compelling form (Burnet 1920). Armstrong calls him the first Greek philosopher who reasoned by logical argument, however primitive it may be (Armstrong 1981). Parmenides tells us about something existent that "It is and it is impossible for it not to be" (Burnet 1920, p. 173). Thus, something non-existent is unthinkable and it is logically contradictory to think about how it looks. Along this line of thought Parmenides brings the ontological and epistemological world together. His One (being) can only be thought of as indivisible. The idea of earlier philosophers that things in some sense "are and are not" must be rejected. Therefore it must also be rejected that something existent changes over time. This originally existing One, being timeless, uniform and necessary, remains equal forever and cannot increase, decrease or be divided. If you are a defender of this One, Plato lets Zeno in discussion with Socrates reason, you cannot at the same time argue in favor of the Many. Then the One would become divisible and, if you continued in that direction, it would lead you to strange paradoxes.

In their studies of Zeno's paradoxes, both Salmon (1980) and Huggett (1999) pointed out that the first step in the solution of the paradox of Achilles and the tortoise was already made by Aristotle. Aristotle questioned Zeno's premise that an infinite amount of time is required to traverse an infinite number of finite lengths and emphasized that if one, length or time, is infinite with respect to divisibility, the other must be as well. One cannot travel an infinitely long distance in a finite amount of time, Aristotle reasoned, but one can traverse an infinite number of parts of a finite distance in a finite amount of time. "Aristotle quite appropriately pointed out that the time span during which Achilles chases the tortoise can likewise be subdivided into infinitely many non-zero intervals, so Achilles has infinitely many non-zero time intervals in which to traverse the infinitely many non-zero space intervals" (Salmon 1980, p. 36). In his attempt to bridge the gap between finite and infinite in this way, Aristotle clearly foreshadowed the limit solution. By emphasizing the similarity between space and time, he also foreshadowed the geometric solution, in which the two are linearly related continuous variables. Although possibly, as seen above, differently scaled.

The paradox indeed goes wrong in interpreting the infinite number of steps in traversing the finite distance as an infinite period of time, but it took almost 2300 years after Aristotle to bridge the gap between finite and infinite by a rigorous definition of limit. Specifically, by clearly defining what infinite addition in the limit solution stands for. The series $S_n = s + vs + \ldots + v^{n-1}s = s\frac{1-v^n}{1-v}$ having the limit $L = s\frac{1}{1-v}$ ($S_n \to L$ for $n \to \infty$), in Cauchy's definition means that for all real numbers $\varepsilon > 0$ there is a $\delta > 0$ such that for all $n > \delta$: $L - S_n < \varepsilon$. Thus, a limit cannot be actually reached but for each conceivable small difference with the limit (ε), we can find sufficiently large numbers ($n > \delta$) that make the difference even smaller. As discussed above, this limit only exists, meaning that such δ can only be found for every ε, in particular for an arbitrarily small ε, if $v < 1$.

4. Granular spacetime and the paradox

4.1. *Are space and time infinitely divisible?*

One big philosophical question remains, which is whether time and space are not only mathematically but also physically infinitely divisible. The answer is no. Already the ancient Greek philosophers speculated about the discreteness of space and time, but the idea has encountered a resurgence in the last 100 years. In his 1925 conference contribution *On the infinite* David Hilbert remarked: “The sort of divisibility needed to realize the infinitely small is nowhere to be found in reality. The infinite divisibility of a continuum is an operation which exists only in thought” (Hilbert 1983, p. 186). Stephen Hawking explained most clearly in his last book, when he talks about the progress in science over the 20th century: “The work on atomic physics in the first thirty years of the century took our understanding down to lengths of a millionth of a millimeter. Since then, research on nuclear and high-energy physics has taken us to length scales that are smaller by a further factor of a billion. It might seem that we could go on forever discovering structures on smaller and smaller length scales. However, there is a limit to this series as with a series of nested Russian dolls. Eventually one gets down to a smallest doll, which can’t be taken apart anymore. In physics the smallest doll is called the Planck length” (Hawking 2018, p. 156).

In the limit solution we indeed allow *n* to increase without bound, so the size of the head starts becomes boundlessly small. It appears that we have to admit that, at least with regard to the divisibility of space, Zeno was right, because according to quantum theory space and time are not infinitely divisible, that is, cannot be split in half ad infinitum. Quantum theory considers space and time to consist of discrete units with the smallest sizes being, respectively, Planck length ($\approx 1.616\times10^{-35}$ meters) and Planck second ($\approx 5.391\times10^{-44}$ seconds). The Planck length is the distance that light travels in one Planck second. As the radius of a hydrogen atom is ‘only’ 2.5×10^{-11} meters, this clarifies how minuscule Planck length and second actually are. To put this into perspective, the following statement concerning relative sizes is approximately true: the hydrogen atom is to the Planck length what the universe is to us (David T. Crouse, personal communication, May 11, 2020) (Crouse 2020).

If space and time are discrete indeed, we must conclude that the paradox solutions presented above are not perfect representations of reality. Because every mathematical model is an approximation of reality, the discrete nature of space and time, intellectually compelling as it is, should not worry us too much. A pertinent example is the law of radioactive decay. The celebrated half-life, the amount of time it takes for half of the atoms to die, relates time to decay by a continuous function. However, the decay necessarily takes place in tiny, discrete steps of individual atoms. Therefore, finally only one atom is left, dying in a last sudden discrete step. Because the quantities of radioactive material found in actual practice are relatively large, there is no reason not to treat and calculate the decay as a continuous process.

What does the discrete or so-called ‘granular’ conception of spacetime mean for the solutions of the paradox? Again, for the macroscopic non-zero distances found in practice no adaptation is required, as the continuous formula for radioactive decay does not need adaptation. The question is relevant, though, from a philosophical standpoint and if one is interested in the study and calculation of extremely small distances with a precision approaching that of the Planck length and second (35 and 44 decimals, respectively). The building blocks of granular spacetime, equaling two times the Planck length and two times the Planck second according to Crouse and Skufca (2019), are called hodon and chronon (Margenau 1949), respectively. The hodon and chronon are sometimes subsumed under the name spason, because both are considered different aspects of a single entity in unified spacetime. The question is what happens when we mathematically meet in continuous spacetime, these smallest particles in granular spacetime.

Efforts have been made over the last 70 years to adapt the continuous algebraic rules to granular spacetime, especially the definition and calculation of distance by the Pythagorean theorem. Crouse and Skufca (2019) explain how the study of granular spacetime has been seriously hampered by the Weyl tile argument (Weyl 1949) and the apparent violations of the laws of special relativity (e.g., length contraction and time dilation). They list and discuss a series of problems formulated with regard to the concept of granular spacetime, also extensively discussed by Hagar (2014), and show how the difficulty of refuting the Weyl tile argument prevented the solution of most of the other problems too. The Weyl tile argument criticized one specific discrete distance formula, proven by Weyl to violate the Pythagorean theorem (hypotenuse being the square root of the sum of the squared triangle sides). Because it did so for all distances, macroscopic as well as microscopic, for quite some time the Weyl tile argument undermined the whole idea of granular spacetime. New discrete distance formulas have since been proposed, however, by Van Bendegem (1987), Forrest (1995) and Crouse (2016). Crouse and Skufca (2019) show that these formulas become identical for conditions that best describe granular spacetime. The results of these formulas differ only from the Pythagorean theorem results for the microscopic distances approaching the Planck length but converge with the Pythagorean theorem results for larger distances found in actual practice.

In granular spacetime, the discrete distance between hodon A and hodon B is defined as the number of jumps of one hodon size, which are needed to cover the path from A to B, that is, to reach B starting from the boundary of A. Calling the sizes of the discrete triangle sides n_1 and n_2, this number m replaces the well-known formula $\sqrt{n_1^2 + n_2^2}$ for the hypotenuse in the continuous case and can be found (Crouse and Skufca 2019, p. 192) by taking the smallest integer m to satisfy the equation

$$m > (\sqrt{n_1^2 + n_2^2} - 1) \tag{14}$$

or for equal triangle sides $n_1 = n_2 = n$

$$m > (\sqrt{2}n - 1) \tag{15}$$

Part of the formula is the continuous hypotenuse formula $\sqrt{n_1{}^2 + n_2{}^2}$ or $\sqrt{2}n$, from which 1 is subtracted. The computation of the discrete hypotenuse size m for different triangle side sizes n is illustrated in Figure 4. A remarkable result is that in the cases of the smallest possible sizes, $n = 1$ and $n = 2$, the discrete hypotenuse turns out to be equal to its triangle side. For increasing triangle sides, $n \geq 3$, the discrete hypotenuse increases in comparison to the triangle sides, similar to that in the continuous case. Most importantly, because the same value 1 is always subtracted, the relative difference of m with the continuous hypotenuse value $\sqrt{2}n$ tends to decrease and becomes negligible for huge values of m.

Crouse and Skufca derived an interesting formula to associate spatial positions in continuous spacetime for arbitrarily chosen axes to positions in granular spacetime (Crouse and Skufca 2019, p. 193):

$$p = (n\chi - l_p, n\chi + l_p] = ((n - 0.5)\chi, (n + 0.5)\chi] \tag{16}$$

l_p is the Planck length, $\chi = 2l_p$ the hodon size and the position in continuous spacetime is given as an interval. This equation leads as follows to distances along a straight line in granular spacetime (David T. Crouse, personal communication, May 11, 2020) (Crouse 2020):

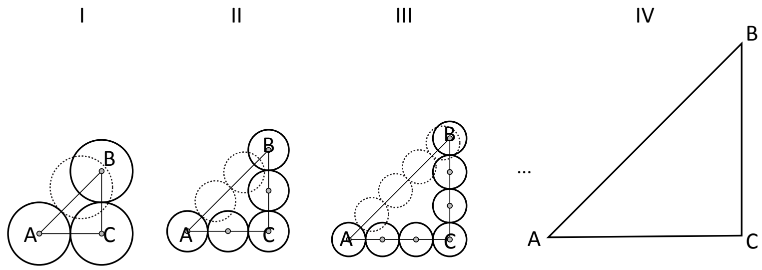
$$d = |p(n_i) - p(n_k)| = |n_k - n_i|\chi \tag{17}$$

Thus, the intervals (each of extent χ) of positions p in continuous spacetime are assigned single values in granular spacetime, and distances in granular spacetime are always integer multiples of χ. No distances smaller than one hodon are possible, which makes it the size of the smallest entity in the universe that can possibly exist. Distances come in multiples of one hodon, while the position interval covers one hodon.

4.2. *Discrete solution*

Again, what does the introduction of granular spacetime mean for the paradox? Most importantly, it provides a third but discrete solution, allowing Achilles to catch up with the tortoise in a finite number of steps. The infinity of steps in reaching the limit is avoided and the paradox is solved in an extremely easy way. Repeating equation (11) for the distances between Achilles and the tortoise (vertical lines in Figure 3) but interpreting it in granular spacetime, we find

$$p_T(i) - p_A(i) = sv^i = n_i\chi \tag{18}$$



n	1	2	3	10000
$\sqrt{2}n$	1.4	2.8	4.2	14142.1
$\sqrt{2}n$-1	0.4	1.8	3.2	14141.1
m	1	2	4	14142

Figure 4: Four examples of discrete distance computation in granular spacetime. The discrete distance or hypotenuse between hodon A and hodon B is defined as the smallest possible number of jumps of hodon size 1 to cover the path from A to B. In the case of equal triangle sides n as shown here, it can be computed as the smallest integer satisfying $m > (\sqrt{2}n - 1)$. Example I shows that for triangle side $n = 1$ (the midpoints of the two hodons A and C as well as B and C are one hodon apart and thus one jump is sufficient to go from A to C and from B to C), one hodon jump from A in the direction of B suffices to partially overlap B, so $m = n = 1$. In example II, two hodon jumps from A enable to partially overlap B, $m = n = 2$, and so again hypotenuse and triangle side are equal. However, in example III with side $n = 3$, giving $\sqrt{2}n - 1 = 3.2$, the hypotenuse is longer than the side: $m = 4$. This is the case for all values $n \geq 3$, similar to that in the continuous case. Finally, example IV clearly illustrates that for huge values of n, in the example 10000, the discrete distance $m = 14142$ is virtually equal to the continuous value $\sqrt{2}n = 14142.1$.

As we have seen in continuous spacetime, for $i \to \infty$, the distance between Achilles and the tortoise converges to zero ($sv^i \to 0$) and $p_T(i)$ and $p_A(i)$ converge to limit $s\frac{1}{1-v}$, solving the paradox in continuous spacetime. However, in granular spacetime, Achilles' position matches that of the tortoise already, as soon as sv^i becomes less than one hodon $sv^i < \chi$. That is the case when we take as the number of steps i the smallest integer satisfying

$$i > \frac{\ln \chi - \ln s}{\ln v} \tag{19}$$

The granular distance n, at which Achilles and the tortoise meet, is then computed as the smallest integer satisfying

$$n > [(s\,\frac{1-v^{i+1}}{1-v})/\chi - 1] \tag{20}$$

Taking chronon as time unit and b as the number of chronons Achilles needs to run one hodon, the elapsed time in chronons is easily calculated as nb.

Let us take the example in Figure 3 with $s = 100$ meters and $v = 0.5$. The number of steps $i = 122$, $s\frac{1-v^{i+1}}{1-v} \approx 200 - 1.881 \times 10^{-35}$ meters and $n \approx 6.188 \times 10^{36}$. Achilles and the tortoise need only 122 steps to meet in granular spacetime, which can be positioned in continuous spacetime at only a tiny fraction before the limit of 200 meters. Because of the rather big s and the extreme smallness of hodon χ the granular distance n at which it takes place is an immensely huge number. But let us replace $s = 100$ meters by the much smaller head start of $s = 8\chi$. Then Achilles would meet the tortoise in 4 steps: $i = 4$, $s\frac{1-v^{i+1}}{1-v} = s + sv + sv^2 + sv^3 + sv^4 = 15.5\chi$, so $n = 15$ and the difference between $s\frac{1-v^{i+1}}{1-v} = 15.5\chi$ and limit $s\frac{1}{1-v} = 16\chi$ is 0.5χ. Starting from the head start s of 8 hodons, the race takes place in packages of, successively, sv of 4 hodons at the first step, sv^2 of 2 hodons at the second step and sv^3 of 1 hodon at the third step. At the 4th step sv^4 of less than one hodon Achilles meets the tortoise and the whole race is over. He skips the infinitude of steps he would still have to run in the remaining distance of 0.5χ in continuous spacetime.

It is clear that the interval of only one hodon in equation (16) becomes relatively smaller with increasing n. The question is when the number of hodons n is sufficiently large to consider the difference between the left and right values in the interval to be negligible, so that we can replace the interval by a single value, the granular approach by well-known continuous spacetime methods and the discrete solution by the continuous spacetime solutions. This depends on how much we allow distance according to the Pythagorean theorem ($\sqrt{n_1^2 + n_2^2}$ or $\sqrt{2}n$ for equal sides $n_1 = n_2 = n$) to deviate from the corresponding value in granular spacetime ($\sqrt{n_1^2 + n_2^2} - 1$ or for equal sides $\sqrt{2}n - 1$). Suppose that we would be satisfied with a relative deviation of 0.001 or smaller, that is $(\sqrt{2}n)/(\sqrt{2}n - 1) \leq 1.001$. This would make the required number of hodons in the discrete distance $n \geq 707$. The solution distance $s\frac{1}{1-v}$, derived above in continuous spacetime for the distance at which Achilles and the tortoise meet, would become correct, at least for all values larger than $707\chi = 1414 l_p \approx 2.285 \times 10^{-32}$. Because $2.285 \times 10^{-32} < 1 \times 10^{-31}$, that is for all values large enough to be expressed as non-zero in 31 or less decimals. As the size of a hydrogen atom needs no more than 12 decimals to be expressed, it is safe to conclude that this solution distance would become correct for most practical purposes. In the example of Figure 3 with $s\frac{1-v^{123}}{1-v}$ virtually indistinguishable from $s\frac{1}{1-v} = 200$ and the granular distance n so extremely huge, sticking to continuous spacetime methods seems quite reasonable indeed.

5. Supertasks, infinity machines and other topics in the recent history of the paradox

"In honor of Zeno, let us apply the name '*Z*-sequence' to an infinite progression of intervals of space or time whose successive magnitude are 1/2, 1/4, 1/8, …, and so on. For the sake of arithmetic simplicity, I shall follow Zeno's procedure and assume that the successive *durations* of the (…) runner's submotions form a *Z*-sequence just as the subintervals of space which are covered by these submotions" (Grünbaum 1955, p. 204). This way, referring to an instance of the more generally formulated sequence in equation (1), Grünbaum started one of his contributions to a heated debate in philosophy that took place from 1950 onward (Black 1950-51, Wisdom 1951-52, Thomson 1954-55, Benacerraf 1962, Vlastos 1966) and still continues. See also the extensive bibliography section in Salmon (2001). It is about the infinite number of steps that Achilles, after completing his race successfully in granular spacetime, would still have to run in continuous spacetime. Such an infinite sequence of acts to be performed in a finite time is called a supertask in the debate. And the discussion focuses on the logical as well as physical, especially kinematic, possibility of performing the supertask. As emphasized by Ardourel in his article on a discrete solution (Ardourel 2015), the supertask problem shows up only in continuous spacetime, that is, only when applying the limit or geometric solution of the paradox. And we can add that the problem resides in a minute part of the finite time interval only, the part covering less than one hodon length (less than 3.23×10^{-35} meters) after completion of the discrete solution's simple task in granular spacetime.

Is performance of a supertask possible? It will be no surprise that most philosophers agree that Achilles in real life is not able to pass or plant an infinitude of flags before reaching the finish flag of his race. Most of the discussion concentrates on whether performing supertasks is logically or kinematically possible. Flag planting instead of or in addition to merely passing was made part of the supertask by Grünbaum (1955). The acts in successive subintervals need sufficient distinctness, "physically individuated motions" in the words of Vlastos (1966, p. 103), to differentiate the 'staccato' stepwise *Z*-sequence run in kinematics from the smooth run in every day life. Because planting a flag takes time, Achilles, himself called legato runner and having equal speed during the entire race, is given a colleague, the staccato runner, who does the planting work. That is, during each subinterval, the staccato runner first runs part of it faster than Achilles, then stops to plant the flag and next makes sure he departs simultaneously with Achilles again at the start of the next subinterval. A further crucial point in the discussion is, what reaching the finish means: approaching the finish flag arbitrarily closely but not touching it or really touching it. That is, whether the interval of the supertask is half open or includes the limit point.

Burke (2000) put the authors about supertasks into three groups:

1. the group of supertask completers who believe in the logical possibility of supertasks,
2. the group of non-believers to which Burke counts himself,
3. and the group of the “reigning orthodoxy”, which accepts some supertasks as logically, kinematically, and dynamically feasible, while other supertasks, if not infeasible logically, are considered at least infeasible kinematically and dynamically.

Burke put Russell and Whitehead in the first group. Russell and Whitehead, although both were impressed by Zeno’s thinking, judged that Zeno missed in his reasoning the fact that an infinite number of finite times may be finite (Russell 1914, p. 49) and that he, out of ignorance of the theory of infinite numerical series, produced an invalid argument (Whitehead 1978, p. 69). By accepting so the validity of the limit solution both Russell and Whitehead became supertask completers.

The non-believers in Burke’s second group with Black and Thomson as most prominent representatives seem to claim that the supertask in Cauchy’s limit approach was a self-contradictory concept. An audacious viewpoint indeed after Cauchy logically impeccably proved that assuming a finite sum for an infinite sequence of finite intervals is not self-contradictory. However, in the opinion of the non-believers the contradiction runs deeper than this and is already present in the sheer expression “infinite series of acts”. To support their claims Black, who coined the term “infinity machines”, and Thomson, who in the title of his 1954-1955 article used the term “supertasks” for the first time, invented several infinity machines. The most well-known and simplest is Thomson’s lamp. This has a single push-button switch on its base. When pushed, the lamp turns on, if off; and turns off, if on. Now the supertask is performed by someone who pushes the switch an infinite number of times, first after 1 minute switched on, then after the next 1/2 minute off, then after another 1/4 minute on again, etc. The sum of this well-known series of time intervals is 2 minutes (limit $s\frac{1}{1-v}$ with $s = 1$ minute and $v = 1/2$). The presumed contradiction is in the state of the lamp after these 2 minutes. It cannot be on, because, if on, it was switched off. It cannot be off, because, if off, it was switched on. So, it must be on and off, or neither on nor off.

It was shown by Benacerraf (1962) and is now generally agreed upon, even by Thomson himself (Thomson 1954-55, p. 131), that the derivations of Black and Thomson were incorrect. A thorough study of supertasks and infinity machines has been done by Grünbaum (1955) in Burke’s third group. Supertasks can be taken to be logically sound in his view, that is without contradictions, if the more complicated case of the staccato run can be proven to be kinematically and thus logically sound. A problematic kinematic aspect of the staccato run is the planting process. Planting requires equal minimal spatial displacements across the *Z*-sequence but within ever decreasing *Z*-subintervals and this in turn would require infinite accelerations. Physicist Richard Friedberg helped to develop a version of the staccato run that obviated this and other problematic features, so that Grünbaum could "conclude without qualification that the staccato run is no less feasible than the legato run and that both are indeed kinematically possible" (Grünbaum 1955, pp. 215-216). Nevertheless, Burke (2000) rejected Grünbaum and Friedberg's staccato run as impossible and claims to have proven that all staccato runs possess features that make them kinematically impossible. Supposing this to be true and even that all supertasks in all possible meanings are physically impossible, this does not mean that they are self-contradictory in the pure logical sense.

Let us take a closer look at Thomson's lamp and why it cannot and does not show contradictory behavior. Thomson assumes that the infinity of switches in the half open interval $[t_0, t_1)$ leads to the lamp being simultaneously on as well as off at time point t_1. However, because the infinity of steps in the supertask occurs in the half open interval, nothing can be said about the state of the lamp at t_1, at least not on the basis of what happened in the interval. Of course, the lamp is on or off at t_1, but that must be the result of something outside. It is important to note that the behavior of the lamp, successively switched on and off, is not a convergent series and therefore its value at t_1 is unpredictable from what happened in the half open interval.

Because his series is supposed to be convergent, the situation of Achilles is fundamentally different. Also in his case the limiting value is not part of the half open interval (Achilles does not touch the finish flag), but is indirectly computable based on it, as the limit of a function approximating this value arbitrarily closely. It is also directly computed at t_1 in the geometric solution, but again not belonging to the half open interval.

More recently a new solution of the paradox and the supertask problem was proposed by McLaughlin and Miller (1992), using internal set theory (Nelson 1977), a branch of nonstandard analysis (Robinson 1966). Whereas 17th and 18th century mathematics was still haunted by the ghostlike infinitesimals (whether they exist or not), which Cauchy in 19th century managed to get rid of by his limit concept, nonstandard analysis made them mathematically respectable again as a new kind of so-called nonstandard numbers. The nonstandard infinitesimal numbers are extremely small, that is, greater than zero but smaller than any standard positive real. Now, a highly interesting theorem of internal set theory asserts the existence of a finite set that, although finite, contains all the standard numbers, including all reals. This enabled McLaughlin and Miller to conclude that a finite set suffices to form

Achilles' '*Z*-sequence' and thus to complete his race. It is clear that for bridging the infinity gap as done in the continuous spacetime solutions, internal set theory may provide an interesting alternative. Not all authors are convinced that it provides an improvement on the continuous spacetime solutions, however. The approach met fierce criticism from Alper and Bridger (1997), who mention as a main point the misleading use of the word 'finite'. Although viewed from inside ('internally') the set is finite indeed, viewed from outside it nevertheless remains infinite. "Indeed, internal set theory, which applies the predicate 'finite' to a set that contains all reals, does not relieve the uneasiness caused by Zeno's paradoxes. It only explains the obscure by the more obscure" (Alper and Bridger 1997, p. 154).

6. Discussion and conclusion

In this article, we discussed extensively three solutions for Zeno's paradox of Achilles and the tortoise. The first one, the limit solution, is the most well-known. It directly follows and answers Zeno's reasoning. However, the second one, the geometric solution, is simpler as well as more informative. It simultaneously yields the exact distance and exact period of time that Achilles needs to catch up with the tortoise. It is unfortunate that the geometric solution receives little or only highly technical attention in the literature (e.g., see Ishikawa 2015). In discussing the philosophical background, we explained how Aristotle made the first step in solving the paradox, a step that finally led to the limit and geometric solutions. The fact, however, that spacetime cannot be broken down beyond the basic units of Planck length and Planck second, was reason to consider granular spacetime and the formulation of the recent discrete solution in granular spacetime.

It is important to note that no inconsistency exists between the solutions in continuous spacetime and the discrete solution. Granular spacetime can be viewed as a restriction on continuous spacetime, saving only what has at least the hodon size, the size of the smallest thing in the universe that can exist. Because only full hodons count in the discrete solution, the number of steps by Achilles and the tortoise in granular spacetime becomes finite and the distance at which Achilles and the tortoise meet slightly shorter than in continuous spacetime. Note also that for the reasons given above the restrictive discrete solution is preferred as a model of reality and the continuous spacetime solutions are seen as approximations of the discrete one, rather precise approximations for most practical purposes though. A method is given to compute the distance for which application of the continuous spacetime methods can still be considered valid.

What then is the relevance of the supertask debate for the solutions considered here? Most of the efforts in the debate aim at proving that examples of supertasks such as the legato and staccato run and infinity machines such as Thomson's lamp are kinematically and logically possible. It is true that kinematically sound supertasks are logically sound as well. But when the status of the supertask-fraught limit solution is simply an expedient approximation of the discrete solution, kinematic soundness is less important. What remains as a requirement is logical soundness in the bare minimum sense of being free from contradiction. There is no proof that any of the solutions discussed would lack the requirement in this minimum sense.

Sometimes a discrete solution is not seen or not only seen as an alternative solution of Zeno's paradox (e.g., Ardourel 2015, Van Bendegem 1987) but as another way to refute his argument. Because space and time are not infinitely divisible, the paradox would miss its point. Of course, we do not know for sure what Zeno has said or written and how he would have reacted to the facts of quantum theory. There are interpretations of Aristotle's text according to which Zeno did not misinterpret the infinite number of steps as an infinite time (Vlastos 1966) but rejected indeed the infinite divisibility of space. If true, it would still enhance Zeno's significance for modern physics and philosophy as recognized earlier by Russell (1914, p. 54). Zeno's reasoning in Plato's dialogue, often celebrated as the first instance of 'reductio ad absurdum' in history, is such that he assumes two possibilities: the One or the Many. He then literally says to Socrates about the motives of his book: "You do not quite catch the motives (…), which was (…) showing that the hypothesis of the existence of the Many involved greater absurdities than the hypothesis of the One" (Plato 1997, 128b-128e). There appears to be little doubt that granularity negates infinite divisibility of space and therefore leads one horn of the dilemma, the Many, to being an absurdity. Therefore, instead of refuting his argument, it would support it and lead to his alternate hypothesis of the One.

What will nowadays be less convincing in Zeno's reasoning for most people is the dichotomy of the One and the Many, and the Many implying infinite divisibility of space. The modern dichotomy is continuous spacetime, which implies infinite divisibility, versus discrete granular spacetime. Rejection of the first indeed leads to the adoption of the second and the validity of a discrete solution of Zeno' paradox, but not to the exclusion of infinitesimal calculus as an extremely useful tool in the study of macroscopic distances in reality.

7. Acknowledgments

We thank David Crouse, Henk Barendregt, and the anonymous reviewers for their helpful comments in the preparation of the manuscript.

References

[1] Alper, J. S. and Bridger, M. (1997), 'Mathematics, models and Zeno's paradoxes', *Synthese* **110**, 143–166.

[2] Ardourel, V. (2015), 'A discrete solution for the paradox of Achilles and the tortoise', *Synthese* **192**(9), 2843–2861.

[3] Aristotle (1957), *The Physics*, Loeb Classical Library edn, Harvard University Press, Cambridge, MA.

[4] Armstrong, A. H. (1981), *An introduction to ancient philosophy*, Littlefield, Adams, Totowa, NJ.

[5] Benacerraf, P. (1962), Tasks, super-tasks, and the modern eleatics, *Journal of Philosophy*, LIX, *in* Salmon (2001), pp. 103–129.

[6] Black, M. (1950-51), Achilles and the tortoise, *Analysis*, XI, *in* Salmon (2001), pp. 67–81.

[7] Burke, M. B. (2000), 'The impossibility of superfeats', *The Southern Journal of Philosophy* **38**(2), 207–220.

[8] Burnet, J. (1920), *Early Greek philosophy*, A & C Black, London.

[9] Crouse, D. and Skufca, J. (2019), 'On the nature of discrete spacetime, Part 1: The distance formula, relativistic time dilation and length contraction in discrete space-time', *Logique et Analyse* **246**, 177–223.

[10] Crouse, D. T. (2016), 'On the nature of discrete space-time: The atomic theory of space-time and its effects on Pythagoras's theorem, time versus duration, inertial anomalies of astronomical bodies, and special relativity at the Planck scale', *arXiv preprint. arXiv:1608.08506* .

[11] Crouse, D. T. (2020), 'No title', Personal Communication.

[12] Forrest, P. (1995), 'Is space-time discrete or continuous? – An empirical question', *Synthese* **103**(3), 327–354.

[13] Grünbaum, A. (1955), Modern science and Zeno's paradoxes of motion, *The Scientific Monthly*, LXXXI, *in* Salmon (2001), pp. 200–250.

[14] Hagar, A. (2014), *Discrete or continuous?: The quest for fundamental length in modern physics*, Cambridge University Press, Cambridge.

[15] Hawking, S. (2018), *Brief answers to big questions*, John Murray Publishers, London.

[16] Hilbert, D. (1983), On the infinite, *in* P. Benacerraf and H. Putnam, eds, 'Philosophy of Mathematics', Cambridge University Press, Cambridge, pp. 183–201.

[17] Huggett, N. (1999), *Space from Zeno to Einstein : classic readings with a contemporary commentary*, MIT Press, Cambridge, MA.

[18] Ishikawa, S. (2015), Linguistic interpretation of quantum mechanics; projection postulate, Technical Report 04, Department of mathematics, Keio University, Keio.

[19] Margenau, H. (1949), 'Marginalia', *American Scientist* **37**(3), 421–427.

[20] McLaughlin, W. I. and Miller, S. L. (1992), 'An epistemological use of nonstandard analysis to answer Zeno's objections against motion', *Synthese* **92**(3), 371–384.

[21] Nelson, E. (1977), 'Internal set theory: a new approach to nonstandard analysis', *Bulletin of the American Mathematical Society* **83**(6), 1165–1198.

[22] Plato (1997), *Plato's Parmenides*, Yale University Press, New Haven.

[23] Robinson, A. (1966), *Non-standard analysis*, 3 edn, North-Holland Publishing Co.
[24] Russell, B. (1914), Our knowledge of the external world, W.W. Norton & Company, New York, Lecture 6: *The problem of infinity considered historically*, *in* Salmon (2001), pp. 45–58.
[25] Salmon, W. C. (1980), *Space, time and motion*, University of Minnesota Press, Minneapolis, MN.
[26] Salmon, W. C., ed. (2001), *Zeno's paradoxes*, Hackett Publishing, Indianapolis, IN.
[27] Thomson, J. F. (1954-55), Tasks and super-tasks, *Analysis*, XV, *in* Salmon (2001), pp. 89–102.
[28] Van Bendegem, J. P. (1987), 'Zeno's paradoxes and the tile argument', *Philosophy of Science* **54**(2), 295–302.
[29] Vlastos, G. (1966), 'Zeno's race course', *Journal of the History of Philosophy* **4**(2), 95–108.
[30] Weyl, H. (1949), *Philosophy of mathematics and natural science*, Princeton University Press, Princeton, NJ.
[31] Whitehead, A. N. (1978), *Process and reality. An Essay in cosmology*, The Free Press, New York.
[32] Wisdom, J. O. (1951-52), 'The hypothesis of cybernetics', *The British Journal for the Philosophy of Science* **2**(5), 1–24.

Theo THEUNISSEN
HAN University of Applied Sciences
Arnhem and Nijmegen, The Netherlands
theo.theunissen@gmail.com

Johan H.L. OUD
Radboud University
Nijmegen, The Netherlands
han.oud@ru.nl